The Universal Teichmüller space, the Siegel Disc and the restricted Grassmannian

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Where am I?

1. P.I. of the FWF Grant "**Banach Poisson-Lie Groups and integrable systems**" at Institut CNRS Pauli (On leave from Lille University)
2. Master Student in **Visual Computing** at TU Wien: *Machine learning, Deep Learning, Baysian Machine Learning, software engineering, 3D-Vision, 3D-reconstruction, Computer Graphics, Visualisation...*
3. On the TU track for **Certification in Diversity** (16 ECTS)

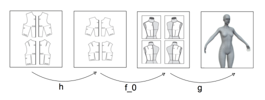
universität wien

wpi
Wolfgang Pauli Institute
c/o Faculty of Mathematics
University of Vienna

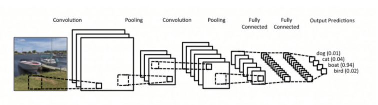
TU WIEN
TECHNISCHE UNIVERSITÄT WIEN

Where am I going?

1. connect the **NLS equation** to **infinite-dimensional Grassmannians**
2. **Garment design:**
   study the dependance of patterns with respect to the stretching properties of fabrics and the curvature of 3D body surface (ABF++ algorithm of Sheffer et al. [?] used in the patternmaking software Modaris from Lectra)



3. **Geometric Pipeline for Machine Learning:**
   - Invariance by group actions
   - metric learning
   - manifold learning
   - frugal learning
   - representation theory



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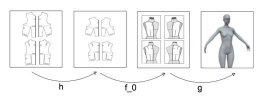

3. **Geometric Green Learning:** GSI 23 Saint-Malo, France
   - Invariance by group actions
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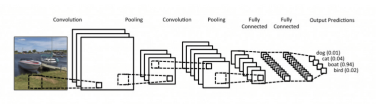

Outline

1. The Universal Teichmüller space
2. the Siegel disc
3. the restricted Grassmannian

Reference :

- L. Guieu, C. Roger, *L'algèbre et le groupe de Virasoro: aspects géométriques et algébriques, généralisations*, Univ. Montréal, 2007.
- F. Gay-Balmaz, T. Ratiu, A.B.Tumpach, *Hyperkähler structures and Universal Teichmüller space*.
- A.B.Tumpach, *Banach Poisson-Lie groups and Bruhat-Poisson structure of the restricted Grassmannian*, Communications in Mathematical Physics, 2020.
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Quasiconformal and quasisymmetric mappings

Definition (quasiconformal mappings)

An orientation preserving homeomorphim f of an open subset A in $\mathbb{C}$ is called quasiconformal if

- f admits distributional derivatives $\partial_z f$, $\partial_{\bar{z}} f \in L^1_{loc}(A, \mathbb{C})$;
- there exists $0 \leq k < 1$ such that $|\partial_{\bar{z}} f(z)| \leq k |\partial_z f(z)|$ for every $z \in A$.

Such an homeomorphism is said to be K-conformal, where $K = \frac{1+k}{1-k}$.

For example, $f(z) = \alpha z + \beta \bar{z}$ with $|\beta| < |\alpha|$ is $\frac{|\alpha|+|\beta|}{|\alpha|-|\beta|}$-quasiconformal.

Quasiconformal and quasisymmetric mappings

Theorem (Lehto 1987)

*An orientation preserving homeomorphism f defined on an open set $A \subset \mathbb{C}$ is **quasiconformal** if and only if it admits distributional derivatives $\partial_z f$, $\partial_{\bar{z}} f \in L^1_{loc}(A, \mathbb{C})$ which satisfy the **Beltrami equation***

$$\partial_{\bar{z}} f(z) = \mu(z) \partial_z f(z), \quad z \in A$$

*for some $\mu \in L^\infty(A, \mathbb{C})$ with $\|\mu\|_\infty < 1$, called the **Beltrami coefficient** or the complex dilatation of f.*

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$.

Theorem (Ahlfors-Bers)

Given $\mu \in L^\infty(\mathbb{D}, \mathbb{C})$ with $\|\mu\|_\infty < 1$, there exists a unique quasiconformal mapping $\omega_\mu : \mathbb{D} \to \mathbb{D}$ with Beltrami coefficient μ, extending continuously to $\overline{\mathbb{D}}$, and fixing $1, -1, i$.

Quasiconformal and quasisymmetric mappings

Definition (quaisymmetric mappings)

An orientation preserving homeomorphism η of the circle $\mathbb{S}^1$ is called **quasisymmetric** if there is a constant $M > 0$ such that for every $x \in \mathbb{R}$ and every $|t| \leq \frac{\pi}{2}$

$$\frac{1}{M} \leq \frac{\tilde{\eta}(x+t) - \tilde{\eta}(x)}{\tilde{\eta}(x) - \tilde{\eta}(x-t)} \leq M,$$

where $\tilde{\eta}$ is the increasing homeomorphism on $\mathbb{R}$ uniquely determined by $0 \leq \tilde{\eta}(0) < 1$, $\tilde{\eta}(x+1) = \tilde{\eta}(x) + 1$, and the condition that its projects onto η.

Theorem (Beurling-Ahlfors extension Theorem)

Let η be an orientation preserving homeomorphism of $\mathbb{S}^1$. Then η is quasisymmetric if and only if it extends to a quasiconformal homeomorphism of the open unit disc $\mathbb{D}$ into itself.

$T(1)$ as a Banach manifold.

Denote by $L^\infty(\mathbb{D})_1$ the unit ball in $L^\infty(\mathbb{D}, \mathbb{C})$.

Definition (The Universal Teichmüller space via Beltrami coefficients)

By Ahlfors-Bers theorem, for any $\mu \in L^\infty(\mathbb{D})_1$, one can consider the unique quasiconformal mapping $w_\mu \ : \mathbb{D} \to \mathbb{D}$ which fixes $-1, -i$ and 1 and satisfies the Beltrami equation on $\mathbb{D}$

$$\frac{\partial}{\partial \overline{z}}\omega_\mu = \mu \frac{\partial}{\partial z}\omega_\mu.$$

Therefore one can define the following equivalence relation on $L^\infty(\mathbb{D})_1$. For μ, $\nu \in L^\infty(\mathbb{D})_1$, set $\mu \sim \nu$ if $w_\mu|\mathbb{S}^1 = w_\nu|\mathbb{S}^1$. The universal Teichmüller space is defined by the quotient space

$$T(1) = L^\infty(\mathbb{D})_1 / \sim .$$

$T(1)$ as a complex Banach manifold.

Theorem (Lehto 1987)

The space $T(1)$ has a unique structure of complex Banach manifold such that the projection map $\Phi : L^\infty(\mathbb{D})_1 \to T(1)$ is a holomorphic submersion.

The differential of Φ at the origin $D_0\Phi : L^\infty(\mathbb{D}) \to T_{[0]}T(1)$ is a complex linear surjection and induces a splitting of $L^\infty(\mathbb{D}, \mathbb{C})$ into [TaTe2004]:

$$L^\infty(\mathbb{D}, \mathbb{C}) = \text{Ker } D_0\Phi \oplus \Omega_\infty(\mathbb{D}, \mathbb{C}),$$

where $\Omega^\infty(\mathbb{D}, \mathbb{C})$ is the Banach space of bounded harmonic Beltrami differentials on $\mathbb{D}$ defined by

$$\Omega_\infty(\mathbb{D}, \mathbb{C}) := \left\{\mu \in L^\infty(\mathbb{D}, \mathbb{C}) \mid \mu(z) = (1 - |z|^2)^2\overline{\phi(z)}, \phi \text{ holomorphic on } \mathbb{D}\right\}.$$

$T(1)$ as a group via quasisymmetric homeomorphisms

T(1) as a group

By the Beurling-Ahlfors extension theorem, a quasiconformal mapping on $\mathbb{D}$ extends to a quasisymmetric homeomorphism on the unit circle leading to the bijection

$$\begin{array}{ccc} T(1) & \stackrel{\sim}{\to} & \mathrm{QS}(\mathbb{S}^1)/PSU(1,1) \\ [\mu] & \mapsto & \left[w_\mu|\mathbb{S}^1\right]. \end{array}$$

- The coset $\mathrm{QS}(\mathbb{S}^1)/PSU(1,1)$ herits from its identification with $T(1)$ a complex Banach manifold structure.
- the coset $\mathrm{QS}(\mathbb{S}^1)/PSU(1,1)$ can be identified with the subgroup of quasisymmetric homeomorphisms fixing $-1, i$ and 1. This identification allows to endow the universal Teichmüller space with a group structure.

Relative to this differential structure, the right translations in $T(1)$ are biholomorphic mappings, whereas the left translations are not even continuous in general. Consequently $T(1)$ is not a topological group.

The WP-metric and the Hilbert manifold structure on $T(1)$.

The Banach manifold $T(1)$ carries a Weil-Petersson metric, which is defined only on a distribution of the tangent bundle [NagVerjovsky1990]. In order to resolve this problem the idea in [TaTe2004] is to change the differentiable structure of $T(1)$.

Theorem (TaTe2004)

The universal Teichmüller space $T(1)$ admits a structure of Hilbert manifold on which the Weil-Petersson metric is a right-invariant strong hermitian metric.

For this Hilbert manifold structure, the tangent space at $[0]$ in $T(1)$ is isomorphic to

$$\Omega_2(\mathbb{D}) := \left\{ \mu(z) = (1 - |z|^2)^2 \overline{\phi(z)}, \quad \phi \text{ holomorphic on } \mathbb{D}, \quad \|\mu\|_2 < \infty \right\},$$

where $\|\mu\|_2^2 = \int\int_{\mathbb{D}} |\mu|^2 \rho(z) d^2 z$ is the L^2-norm of μ with respect to the hyperbolic metric of the Poincaré disc $\rho(z) d^2 z = 4(1 - |z|^2)^{-2} d^2 z$.

The WP-metric and the Hilbert manifold structure on $T(1)$.

Definition (Weil-Petersson metric on $T(1)$)

The Weil-Petersson metric on $T(1)$ is given at the tangent space at $[0] \in T(1)$ by

$$\langle \mu, \nu \rangle_{WP} := \iint_{\mathbb{D}} \mu \, \overline{\nu} \, \rho(z) d^2 z$$

With respect to its Hilbert manifold structure, $T(1)$ admits uncountably many connected components. For this Hilbert manifold structure, the identity component $T_0(1)$ of $T(1)$ is a topological group.

The WP-metric and Virasoro coadjoint orbit.

T(1) and Virasoro coadjoint orbit

The pull-back of the Weil-Petersson metric on the quotient space $\mathrm{Diff}_+(\mathbb{S}^1)/\operatorname{PSU}(1,1) \subset QS(\mathbb{S}^1)/\operatorname{PSU}(1,1)$ is given at $[\mathrm{Id}]$ by

$$h_{WP}([\mathrm{Id}])([u],[v]) = 2\pi \sum_{n=2}^{\infty} n(n^2-1)u_n\overline{v_n}.$$

Hence $T_0(1)$ of $T(1)$ can be seen as the completion of $\mathrm{Diff}_+(\mathbb{S}^1)/\operatorname{PSU}(1,1)$ for the $H^{3/2}$-norm.

This metric make $T(1)$ into a strong Kähler-Einstein Hilbert manifold, with respect to the complex structure given at $[\mathrm{Id}]$ by the Hilbert transform. The tangent space at $[\mathrm{Id}]$ consists of Sobolev class $H^{3/2}$ vector fields modulo $\mathfrak{psu}(1,1)$.

The WP-metric and Virasoro coadjoint orbit.

T(1) and Virasoro coadjoint orbit

The associated Riemannian metric is given by

$$g_{WP}([\mathrm{Id}])([u],[v]) = \pi \sum_{n\neq -1,0,1} |n|(n^2-1)u_n\overline{v_n},$$

and the imaginary part of the Hermitian metric is the two-form

$$\omega_{WP}([\mathrm{Id}])([u],[v]) = -i\pi \sum_{n\neq -1,0,1} n(n^2-1)u_n\overline{v_n}.$$

Note that ω_{WP} coincides with the Kirillov-Kostant-Souriau symplectic form obtained on $\mathrm{Diff}_+(\mathbb{S}^1)/\,\mathrm{PSU}(1,1)$ when considered as a coadjoint orbit of the Bott-Virasoro group.

The Universal Teichmüller space and Shapes in the plane

The fingerprint map

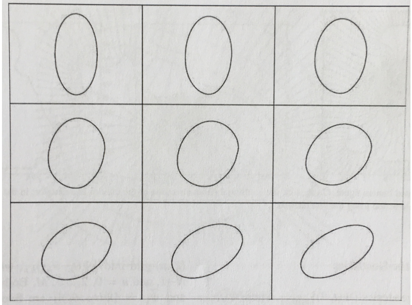
Consider a Jordan curve γ in the plane. Denote by $\mathscr{O}$ and $\mathscr{O}^*$ the two connected components of $\mathbb{C} \setminus \gamma$. By the Riemann mapping theorem, there exists two conformal maps $f : \mathbb{D} \to \mathscr{O}$ and $g : \mathbb{D}^* \to \mathscr{O}^*$ from the unit disc $\mathbb{D}$ and $\mathbb{D}^* := \{z \in \mathbb{C}, |z| > 1\}$ into $\mathscr{O}$ and $\mathscr{O}^*$ respectively. Both f and g extends to homeomorphisms between the closure of the domains and one can form the **conformal welding**

$$h := g^{-1} \circ f : \mathbb{S}^1 \to \mathbb{S}^1.$$

There exists homeomorphisms of $\mathbb{S}^1$ that are not conformal weldings, but any quasi-symmetric homeomorphism h is a conformal welding and the decomposition $h = g^{-1} \circ f$, where $f : \mathbb{D} \to \mathscr{O}$ and $g : \mathbb{D}^* \to \mathscr{O}^*$ are conformal, is unique. The Jordan curve $\gamma := f(\mathbb{S}^1) = g(\mathbb{S}^1)$ is called the **quasi-circle** associated to h. Reciprocally, the map that to a quasi-circle associates a quasi-symmetric homeomorphism of $\mathbb{S}^1$ is called the **fingerprint map** of γ.

The Universal Teichmüller space and Shapes in the plane

Using the fingerprint map, we can pull-back the Weil-Petersson metric of $\mathrm{QS}(\mathbb{S}^1)/PSU(1,1) = T(1)$ to the set of quasi-circles modulo translations and scaling. The geodesics between quasi-circles for the Weil-Petersson metric fournish interpolations between 2D-contours in the plane [SharonMumford2006].



The Siegel disc.

Let $\mathscr{V} = H^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R})/\mathbb{R}$ be the Hilbert space of real valued $H^{\frac{1}{2}}$ functions with mean-value zero. The real Hilbert inner product on $\mathscr{V}$ is given by

$$\langle u, v \rangle_{\mathscr{V}} = \sum_{n\in\mathbb{Z}} |n| u_n \overline{v_n}.$$

Endow the real Hilbert space $\mathscr{V}$ with the following complex structure (called the Hilbert transform)

$$\mathrm{J}\left(\sum_{n\neq 0} u_n e^{inx}\right) = i \sum_{n\neq 0} \operatorname{sgn}(n) u_n e^{inx}.$$

Now $\langle \cdot, \cdot \rangle_{\mathscr{V}}$ and J are compatible in the sense that J is orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathscr{V}}$. The associated (real) symplectic form is defined by

$$\Omega(u, v) = \langle u, J(v) \rangle_{\mathscr{V}} = \frac{1}{2\pi} \int_{\mathbb{S}^1} u(x) \partial_x v(x) dx = -i \sum_{n\in\mathbb{Z}} n u_n \overline{v_n}.$$

The Siegel disc.

Let us consider the **complexified Hilbert space** $\mathscr{H} := H^{1/2}(\mathbb{S}^1, \mathbb{C})/\mathbb{C}$ and the complex linear extensions of J and Ω still denoted by the same letters. Each element $u \in \mathscr{H}$ can be written as

$$u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx} \quad \text{with} \quad u_0 = 0 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} |n||u_n|^2 < \infty.$$

The eigenspaces $\mathscr{H}_+$ and $\mathscr{H}_-$ of the operator J are

$$\mathscr{H}_+ = \left\{ u \in \mathscr{H} \;\middle|\; u(x) = \sum_{n=1}^{\infty} u_n e^{inx} \right\}$$
$$\mathscr{H}_- = \left\{ u \in \mathscr{H} \;\middle|\; u(x) = \sum_{n=-\infty}^{-1} u_n e^{inx} \right\},$$

and one has the Hilbert decomposition $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ into the sum of closed orthogonal subspaces.

The Siegel disc

Definition (The Siegel disc)

The Siegel disc associated with $\mathscr{H}$ is defined by

$$\mathfrak{D}(\mathscr{H}) := \{Z \in L(\mathscr{H}_-, \mathscr{H}_+) \mid Z^T = Z, \forall\, u, v \in \mathscr{H}_- \quad \text{and} \quad I - Z\bar{Z} > 0\}.$$

The restricted Siegel disc associated with $\mathscr{H}$ is by definition

$$\mathfrak{D}_{\mathrm{res}}(\mathscr{H}) := \{Z \in \mathfrak{D}(\mathscr{H}) \mid Z \in L^2(\mathscr{H}_-, \mathscr{H}_+)\}.$$

The restricted Siegel disc as an homogeneous space.

Symplectic group and its restricted version

Consider the symplectic group $\mathrm{Sp}(\mathscr{V}, \Omega)$ of bounded linear maps on $\mathscr{V}$ which preserve the symplectic form Ω

$$\mathrm{Sp}(\mathscr{V}, \Omega) = \{a \in \mathrm{GL}(\mathscr{V}) \mid \Omega(au, av) = \Omega(u, v), \ \text{ for all } u, v \in \mathscr{V}\}.$$

The restricted symplectic group $\mathrm{Sp}_{\mathrm{res}}(\mathscr{V}, \Omega)$ is

$$\mathrm{Sp}_{\mathrm{res}}(\mathscr{V}, \Omega) := \left\{ \left(\begin{array}{cc} g & h \\ \bar{h} & \bar{g} \end{array} \right) \in \mathrm{Sp}(\mathscr{V}, \Omega) \middle| \, h \in L^2(\mathscr{H}_-, \mathscr{H}_+) \right\}.$$

The restricted Siegel disc as an homogeneous space.

Theorem

The restricted symplectic group acts transitively on the restricted Siegel disc by

$$\mathrm{Sp}_{\mathrm{res}}(\mathscr{V}, \Omega) \times \mathfrak{D}_{\mathrm{res}}(\mathscr{H}) \longrightarrow \mathfrak{D}_{\mathrm{res}}(\mathscr{H}),$$

$$\left(\begin{pmatrix} g & h \\ \bar{h} & \bar{g} \end{pmatrix}, Z \right) \longmapsto (gZ + h)(\bar{h}Z + \bar{g})^{-1}.$$

The isotropy group of $0 \in \mathfrak{D}_{\mathrm{res}}(\mathscr{H})$ *is the unitary group* $\mathrm{U}(\mathscr{H}_+)$ *of* $\mathscr{H}_+$*, and the restricted Siegel disc is diffeomorphic as Hilbert manifold to the homogeneous space*

$$\mathrm{Sp}_{\mathrm{res}}(\mathscr{V}, \Omega) / \mathrm{U}(\mathscr{H}_+).$$

The Siegel disc as a generalization of the Poincaré disc

In the previous construction, replace $\mathscr{V}$ by $\mathbb{R}^2$ endowed with its natural symplectic structure. The corresponding Siegel disc is nothing but the open unit disc $\mathbb{D}$. The action of $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$ is the standard action of $\mathrm{SU}(1,1)$ on $\mathbb{D}$ given by

$$z \in \mathbb{D} \longmapsto \frac{az+b}{\bar{b}z+\bar{a}} \in \mathbb{D}, \quad |a|^2 - |b|^2 = 1,$$

and the Hermitian metric obtained on $\mathbb{D}$ is given by the hyperbolic metric

$$h_{\mathfrak{D}}(z)(u,v) = \frac{1}{(1-|z|^2)^2} u\bar{v}.$$

Therefore, $\mathfrak{D}_{\mathrm{res}}(\mathscr{H})$ can be seen as an infinite-dimensional generalization of the Poincaré disc.

Period mapping

Theorem (Theorem 3.1 in NagSullivan1995)

For φ a orientation preserving homeomorphism and any $f \in \mathscr{V} = H^{\frac{1}{2}}(\mathbb{S}^1, \mathbb{R})/\mathbb{R}$, define

$$V_\varphi f = f \circ \varphi - \frac{1}{2\pi} \int_{\mathbb{S}^1} f \circ \varphi.$$

Then V_φ maps $\mathscr{V}$ into itself iff φ is quasisymmetric.

Theorem (Proposition 4.1 in NagSullivan1995)

The group $\mathrm{QS}(\mathbb{S}^1)$ of quasisymmetric homeomorphisms of the circle acts on the right by symplectomorphisms on $\mathscr{V}$ by

$$V_\varphi f = f \circ \varphi - \frac{1}{2\pi} \int_{\mathbb{S}^1} f \circ \varphi,$$

$\varphi \in \mathrm{QS}(\mathbb{S}^1)$, $f \in \mathscr{V}$.

Period mapping

Theorem (Theorem 7.1 in NagSullivan1995)

This action defines a map $\Pi : \mathrm{QS}(\mathbb{S}^1) \to \mathrm{Sp}(\mathscr{V}, \Omega)$. *The operator* $\Pi(\varphi)$ *preserves the subspaces* $\mathscr{H}_+$ *and* $\mathscr{H}_-$ *iff* φ *belongs to* $\mathrm{PSU}(1,1)$. *The resulting map is an injective equivariant holomorphic immersion*

$$\Pi \; : T(1) = \mathrm{QS}(\mathbb{S}^1)/\,\mathrm{PSU}(1,1) \to \mathrm{Sp}(\mathscr{V}, \Omega)/\,\mathrm{U}(H_+) \simeq \mathfrak{D}(\mathscr{H})$$

called the ***period mapping*** *of* $T(1)$.

The Hilbert version of the period mapping is given by the following

Theorem (TaTe2004)

For $[\mu] \in T(1)$, $\Pi([\mu])$ *belongs to the restricted Siegel disc if and only if* $[\mu] \in T_0(1)$. *Moreover the pull-back of the natural Kähler metric on* $\mathfrak{D}_{\mathrm{res}}(\mathscr{H})$ *coincides, up to a constant factor, with the Weil-Petersson metric on* $T_0(1)$.

Polarizations

Definition (Polarizations)

A **polarization** of $\mathscr{H}$ is a complex and closed subspace W of $\mathscr{H}$ such that

$$\mathscr{H} = W \oplus \overline{W},$$

where $\overline{W} = \{\bar{w} \in \mathscr{H} \mid w \in W\}$.

Note that, since Ω is extended to $\mathscr{H}$ by complex bilinearity, we have

$$\overline{\Omega(w, z)} = \Omega(\bar{w}, \bar{z}).$$

Therefore

$$\overline{i\Omega(w, \bar{w})} = -i\Omega(\bar{w}, w) = i\Omega(w, \bar{w}).$$

This proves that $i\Omega(w, \bar{w}) \in \mathbb{R}$.

Polarizations

Definition (Positive polarizations)

A **positive polarization relative to** Ω is a polarization of $\mathscr{H}$ such that

$$i\Omega(w, \bar{w}) > 0, \quad \text{for all } w \in W, w \neq 0.$$

Definition (Positive isotropic polarizations)

We denote by $\mathscr{P}ol(\mathscr{H}, \Omega)$ the set of all isotropic polarizations of $\mathscr{H}$ and by $\mathscr{P}ol^{+}(\mathscr{H}, \Omega)$ the set of all positive and isotropic polarizations of $(\mathscr{H}, \Omega)$.

Complex structures

Definition (Complex structures compatible with a symplectic form)

Given a real Hilbert space $(\mathcal{V}, \Omega)$ endowed with a strong symplectic form, a complex structure $K : \mathcal{V} \to \mathcal{V}$, $K^2 = -I$ is said to be **compatible with** Ω if

$$\Omega(Kw, Kz) = \Omega(w, z)$$

for all w, z in $\mathcal{V}$. We denote by $\mathscr{J}(\mathcal{V}, \Omega)$ the set of all **linear complex structures on** $\mathcal{V}$ **compatible** with Ω.

Given $K \in \mathscr{J}(\mathcal{V}, \Omega)$ we can form the symmetric and strongly nondegenerate bilinear form on $\mathcal{V}$

$$g_K(u, v) := \Omega(Ku, v).$$

We say that K is **positive** if $g_K(u, u) > 0 \forall u \neq 0$. We denote by $\mathscr{J}^+(\mathcal{V}, \Omega)$ the set of all compatible positive complex structures on $\mathcal{V}$.

The restricted Grassmannian

Definition (The restricted Grassmannian Gr_{res})

A closed subspace W of $\mathscr{H}$ belongs to the restricted Grassmannian Gr_{res} iff

1. $p_- \; : W \to \mathscr{H}_-$ is Hilbert-Schmidt,
2. $p_+ \; : W \to \mathscr{H}_+$ is Fredholm

The restricted Grassmannian as homogeneous space

$$GL_{res} = \{\left\{ \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right\} \in GL(\mathscr{H}), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$P_{res} = \{\left\{ \begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix} \right\} \in GL(\mathscr{H}), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$U_{res} = \{\left\{ \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right\} \in U(\mathscr{H}), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$Gr_{res} \simeq GL_{res}/P_{res} \simeq U_{res}/(U(\mathscr{H}_+) \times U(\mathscr{H}_-))$$

Injection of the Siegel disc into the restricted Grassmannian

Theorem (GBT)

The action of $\mathsf{GL}_{\mathrm{res}}(\mathscr{H})$ *on* $\mathsf{Gr}_{\mathrm{res}}(\mathscr{H})$ *restricts to a transitive action of* $\mathsf{Sp}_{\mathrm{res}}(\mathscr{V},\Omega)$ *on* $\mathscr{P}ol^{+}_{\mathrm{res}}(\mathscr{H},\Omega)$*. The isotropy group of* $\mathscr{H}_{+}$ *is*

$$\left\{\begin{pmatrix} g & 0 \\ 0 & \bar{g}\end{pmatrix} \in \mathsf{Sp}_{\mathrm{res}}(\mathscr{V},\Omega)\right\} \simeq \mathsf{U}(\mathscr{H}_{+}),$$

and we have isomorphisms:

$$\mathsf{Sp}_{\mathrm{res}}(\mathscr{V},\Omega)/\,\mathsf{U}(\mathscr{H}_{+}) \simeq \mathfrak{D}_{\mathrm{res}}(\mathscr{H}) \simeq \mathscr{P}ol^{+}_{\mathrm{res}}(\mathscr{H},\Omega) \simeq \mathscr{J}^{+}_{\mathrm{res}}(\mathscr{V},\Omega)$$

as well as an holomorphic injection

$$\mathfrak{D}_{\mathrm{res}}(\mathscr{H}) \simeq \mathsf{Sp}_{\mathrm{res}}(\mathscr{V},\Omega)/\,\mathsf{U}(\mathscr{H}_{+}) \hookrightarrow GL_{res}/P_{res} \simeq Gr_{res}$$

What are the traps of infinite-dimensional geometry?

In infinite-dimensional geometry, the golden rule is :
"Never believe anything you have not proved yourself!"

- The distance function associated to a Riemannian metric may by the zero function (for example for the Virasoro group endowed with right invariant L^2-metric).
- Levi-Cevita connection may not exist for weak Riemannian metrics
- Hopf-Rinow Theorem does not hold in general : geodesic completeness $\neq$ metric completeness
- Darboux Theorem does not apply to weak symplectic forms
- A formally integrable complex structure does not imply the existence of a holomorphic atlas
- the tangent space differs from the space of derivations (even on a Hilbert space)
- a Poisson bracket may not be given by a bivector field (even on a Hilbert space)
- there are Lie algebras that can not be enlarged to Lie groups (Examples by Milnor or Neeb)

Finite-dimensional Poisson–Lie groups

References for finite-dimensional Poisson-Lie groups

V.G. Drinfel'd, '83
Y. Kosmann-Schwarzbach, F. Magri, '88
J.-H. Lu, '91

Poisson–Lie groups in the finite-dimensional case

Manin triples

$\Updownarrow$

Lie-bialgebras

$\Updownarrow$

connected simply connected Poisson–Lie groups

Poisson–Lie groups

Let us start with an example of a Manin triple...

$\mathfrak{u}(n)$ = Lie-algebra of the unitary group $\mathrm{U}(n)$
= space of skew-symmetric matrices

$\mathfrak{b}(n)$ = Lie-algebra of the Borel group $\mathrm{B}(n, \mathbb{C})$
= space of upper triangular matrices with real coef. on diagonal

Then the space $M(n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})$ of all complex matrices decomposes :

$$M(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$$

and for the non-degenerate symmetric bilinear continuous map $\langle \cdot, \cdot \rangle$ given by

$$\langle A, B \rangle = \mathrm{Im}\,\mathrm{Tr}(AB) = \text{ imaginary part of trace}(AB)$$

the blocks $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$ are both isotropic.

Poisson–Lie groups

Definition of a Manin triple

A **Banach Manin** triple consists of a triple of Banach Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ over a field $\mathbb{K}$ and a **non-degenerate symmetric bilinear** continuous map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ such that

1. the bilinear map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is invariant with respect to the bracket $[\cdot, \cdot]_{\mathfrak{g}}$ of $\mathfrak{g}$, i.e.

$$\langle [x, y]_{\mathfrak{g}}, z \rangle_{\mathfrak{g}} + \langle y, [x, z]_{\mathfrak{g}} \rangle_{\mathfrak{g}} = 0, \quad \forall x, y, z \in \mathfrak{g}; \tag{1}$$

2. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Banach spaces;
3. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Banach Lie subalgebras of $\mathfrak{g}$;
4. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to the bilinear map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

Let M be a finite-dimensional manifold.

Poisson bracket

A **Poisson bracket** on M is a bilinear map $\{\cdot,\cdot\} : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ with

- skew-symmetry $\{f,g\} = -\{g,f\}$
- Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$
- Leibniz rule $\{f,gh\} = \{f,g\}h + g\{f,h\}$

Poisson tensor

$\{f,g\} = \pi(df,dg)$ where $\pi \in \Gamma(\Lambda^2 TM)$ is a bivector field

Example

Any symplectic manifold is a Poisson manifold

Let G be a finite-dimensional Lie group.

Poisson-Lie groups

A **Poisson-Lie group** G is a Lie group equipped with a Poisson structure compatible with the group multiplication.

Example

Any Lie group G with $\{\cdot,\cdot\} = 0$ is a Poisson Lie group
Any compact Lie group, like $SU(n)$, is a Poisson-Lie group in a non-trivial way.

Definition

For any Poisson-Lie group (G,π), with Lie algebra $\mathfrak{g}$, one defines $\Pi_r^G := R_g^*\pi \ : G \to \Lambda^2\mathfrak{g}$ as

$$\Pi_r^G(g)(\alpha,\beta) := \pi(R_g^*\alpha, R_g^*\beta), \quad \alpha,\beta \in \mathfrak{g}^*$$

Let (G, π) be a finite-dimensional Poisson-Lie group.

Facts

1. the fact that the Poisson tensor π is compatible with the group multiplication implies the following cocycle condition on $\Pi_r^G := R_g^* \pi$

$$\Pi_r^G(gh) = \Pi_r^G(g) + \mathrm{Ad}_g \Pi_r^G(h)$$

2. the derivative of $\Pi_r^G \ : G \to \Lambda^2\mathfrak{g}$ at the unit of the group is a cocycle $\theta \ : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$

$$\begin{aligned} \theta\left([x, y]\right)(\alpha, \beta) = \ & \theta(y)(\mathrm{ad}_x^* \alpha, \beta) + \theta(y)(\alpha, \mathrm{ad}_x^* \beta) \\ & - \theta(x)(\mathrm{ad}_y^* \alpha, \beta) - \theta(x)(\alpha, \mathrm{ad}_y^* \beta) \end{aligned}$$

where $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$.

3. the Jacobi identity verified by the Poisson structure implies that $\theta^* := [\cdot, \cdot]_{\mathfrak{g}}^* \ : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$

Finite-dimensional Lie bialgebras

Lie bialgebra

Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$. One says that $(\mathfrak{g}, \mathfrak{g}^*)$ form a **Lie bialgebra** if there is a Lie bracket $\Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ whose dual map $\mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$

$$\begin{aligned}\theta\left([x,y]\right)(\alpha,\beta) = \; & \theta(y)(\mathrm{ad}^*_x\alpha,\beta) + \theta(y)(\alpha,\mathrm{ad}^*_x\beta) \\ & -\theta(x)(\mathrm{ad}^*_y\alpha,\beta) - \theta(x)(\alpha,\mathrm{ad}^*_y\beta)\end{aligned}$$

Poisson–Lie groups in the finite-dimensional case

Manin triples

$\Updownarrow$

Lie-bialgebras

$\Updownarrow$

connected simply connected Poisson–Lie groups

Example

- $M(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$ with $\langle A, B\rangle = \mathrm{Im}\ \mathrm{Tr} AB$ is a Manin triple.
- $\mathrm{U}(n)$ and $\mathrm{B}(n,\mathbb{C})$ are dual Poisson-Lie groups with

$$\Pi_r^G(g)(x_1, x_2) = \Im \mathrm{Tr}\, p_{\mathfrak{u}}(g^{-1}x_1 g) p_{\mathfrak{b}}(g^{-1}x_2 g).$$

- Moreover $GL(n,\mathbb{C}) = U(n) \times B(n)$ because of Iwasawa dec.
- This gives a dressing action

$$\varphi : B(n) \times U(n) \to U(n)$$

by $\varphi(b)(k) = k'$ where k' is the unique element of $U(n)$ such that $bk = k'b'$ with $b' \in B(n)$.

Reference :

J.-H. Lu, A. Weinstein, *Poisson Lie groups, Dressing Transformations, and Bruhat Decompositions*, Journal of Differential Geometry, 1990.

Poisson manifold modelled on a non-separable Banach space

Problems :

(1) no bump functions available (norm not even $\mathscr{C}^1$ away from the origin)

(2) there exist derivations of order greater then 1 [Kriegl, Michor, '97]

(3) there exist Poisson bracket without Poisson tensor (Leibniz rule does not imply existence of Poisson tensor) [Beltita, Golinski, T., 2018]

(4) existence of Hamiltonian vector field is not automatic

References for Poisson geometry on Banach manifods

A. A. Odzijewicz, T. Ratiu, 2003
P. Cabau, F. Pelletier, 2011
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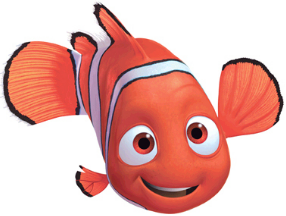
D. Beltita, T. Golinski, A.B.Tumpach, 2018

Poisson bracket not given by a Poisson tensor

Queer Poisson Bracket = Poisson bracket not given by a Poisson tensor



Reference :

D. Beltiţă, T. Goliński, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics, 2018.

Definition of a Banach Poisson manifold

Definition of a Poisson tensor :
M Banach manifold, $\mathbb{F}$ a subbundle of T^*M in duality with TM.
π smooth section of $\Lambda^2\mathbb{F}^*(\mathbb{F})$ is called a Poisson tensor on M with respect to $\mathbb{F}$ if :

1. for any closed local sections α, β of $\mathbb{F}$, the differential $d\left(\pi(\alpha, \beta)\right)$ is a local section of $\mathbb{F}$;
2. (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\left(\alpha, d\left(\pi(\beta, \gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma, \alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha, \beta)\right)\right) = 0.$$

Definition of a Poisson Manifold :
A Banach Poisson manifold is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Banach symplectic manifold

Any Banach symplectic manifold (M, ω) is naturally a generalized Banach Poisson manifold $(M, \mathbb{F}, \pi)$ with

1. $\mathbb{F} = \omega^\sharp(TM)$;
2. $\pi \; : \omega^\sharp(TM) \times \omega^\sharp(TM) \to \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \omega(X_\alpha, X_\beta)$ where X_α and X_β are uniquely defined by $\alpha = \omega(X_\alpha, \cdot)$ and $\beta = \omega(X_\beta, \cdot)$.

Definition

Consider a duality pairing $\langle \cdot, \cdot \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} \; : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$ between 2 Banach. $\mathfrak{g}_+$ is a **Banach Lie–Poisson space with respect to** $\mathfrak{g}_-$ if

- $\mathfrak{g}_-$ is a Banach Lie algebra $(\mathfrak{g}_-, [\cdot, \cdot]_{\mathfrak{g}_-})$
- $\mathfrak{g}_-$ acts continuously on $\mathfrak{g}_+ \hookrightarrow \mathfrak{g}_-^*$ by coadjoint action, i.e.

$$\mathrm{ad}^*_\alpha x \in \mathfrak{g}_+,$$

for all $x \in \mathfrak{g}_+$ and $\alpha \in \mathfrak{g}_-$, and $\mathrm{ad}^* \; : \mathfrak{g}_- \times \mathfrak{g}_+ \to \mathfrak{g}_+$ is continuous.

Banach Poisson-Lie groups

A **Banach Poisson-Lie group** B is a Banach Lie group equipped with a Banach Poisson manifold structure compatible with the multiplication

Proposition

Let B be a Banach Lie group and $(B, \mathbb{B}, \pi)$ a Banach Poisson structure on B. Then B is a Banach Poisson-Lie group if and only if

1. $\mathbb{B}$ is invariant under left and right multiplications by elements in B,
2. the subspace $\mathfrak{u} := \mathbb{B}_e \subset \mathfrak{b}^*$, where e is the unit element of B, is invariant under the coadjoint action of B on $\mathfrak{b}^*$ and the map

$$\begin{array}{rccc} \Pi_r^B : & B & \to & \Lambda^2\mathfrak{u}^* \\ & g & \mapsto & R_{g^{-1}}^{**}\pi_g, \end{array}$$

is a 1-cocycle on B with respect to the coadjoint representation of B in $\Lambda^2\mathfrak{u}^*$.

Theorem [T] :

Let $(G_+, \mathbb{F}, \pi)$ be a Banach Poisson–Lie group. Then $\mathfrak{g}_+$ is a Banach Lie bialgebra with respect to $\mathfrak{g}_-$. The Lie bracket in $\mathfrak{g}_-$ is given by

$$[\alpha_1, \beta_1]_{\mathfrak{g}_-} := T_e\Pi_r(\cdot)(\alpha_1, \beta_1) \in \mathfrak{g}_- \subset \mathfrak{g}_+^*, \quad \alpha_1, \beta_1 \in \mathfrak{g}_- \subset \mathfrak{g}_+^*, \qquad (2)$$

where $\Pi_r := R_{g^{-1}}^{**}\pi : G_+ \to \Lambda^2\mathfrak{g}_-^*$, and $T_e\Pi_r : \mathfrak{g}_+ \to \Lambda^2\mathfrak{g}_-^*$ denotes the differential of Π_r at the unit element $e \in G_+$.

Theorem [T] :

Let $(G_+, \mathbb{F}, \pi)$ be a Banach Poisson–Lie group.If the map $\pi^\sharp : \mathbb{F} \to \mathbb{F}^*$ defined by $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in $TG_+ \subset \mathbb{F}^*$, then $\mathfrak{g}_+$ is a Banach Lie–Poisson space with respect to $\mathfrak{g}_- := \mathbb{F}_e$.

Poisson–Lie groups in the infinite-dimensional case

Manin triple

$\Updownarrow$

Banach Lie-bialgebra + Banach Lie-Poisson space

$\Uparrow$

Banach Poisson–Lie group G + $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in TG

$\mathscr{H}$ separable infinite-dimensional Hilbert space.
On bounded operators $A \in L(\mathscr{H})$ acting on $\mathscr{H}$, define

$$\|A\|_p = \left(\mathrm{Tr}(A^*A)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

For $1 < p < 2 < q < +\infty$, one has:

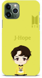
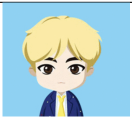
$$L^1(\mathscr{H}) \subset L^p(\mathscr{H}) \subset L^2(\mathscr{H}) \subset L^q(\mathscr{H}) \subset L(\mathscr{H})$$

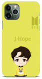
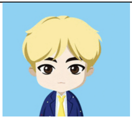
For a decomposition, $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$,

$\mathrm{U}_{res}(\mathscr{H}) = \{\left\{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right\} \in U(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$

$\mathrm{U}_{1,2}(\mathscr{H}) = \{\left\{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right\} \in U(H), A \text{ and } D \text{ Trace-class}, B \text{ and } C \in L^2(\mathscr{H})\}$

$\mathrm{L}_{1,2}(\mathscr{H}) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), A \text{ and } C \text{ Trace class}, B \text{ and } C \text{ Hilbert-Schmidt}\}$

				
Double Lie group	Group	Lie algebra $\mathfrak{g}$	Fiber $\mathbb{F}_e \subset \mathfrak{g}^*$	Dual Group
$GL(n, \mathbb{C})$	$U(n)$	$\mathfrak{u}(n)$	$\mathfrak{b}(n)$	$B(n)$
$GL_2(\mathscr{H})$	$U_2(\mathscr{H})$	$\mathfrak{u}_2(\mathscr{H})$	$\mathfrak{b}_2(\mathscr{H})$	$B_2(\mathscr{H})$
$GL_p(\mathscr{H}), 1 < p \leq 2$	$U_p(\mathscr{H})$	$\mathfrak{u}_p(\mathscr{H})$	$\mathfrak{b}_p(\mathscr{H})$	$B_p(\mathscr{H})$

				
Double Lie group	Group	Lie algebra $\mathfrak{g}$	Fiber $\mathbb{F}_e \subset \mathfrak{g}^*$	Dual Group
???	$U(\mathcal{H})$	$\mathfrak{u}(\mathcal{H})$	$L^1(\mathcal{H})/\mathfrak{u}_1$	???
???	$U_{res}(\mathcal{H})$	$\mathfrak{u}_{res}(\mathcal{H})$	$L^{1,2}(\mathcal{H})/\mathfrak{u}_{1,2}$	???
???	$U_1(\mathcal{H})$	$\mathfrak{u}_1(\mathcal{H})$	$L^1(\mathcal{H})/\mathfrak{u}_1$	???

Example of bounded operator with unbounded triangular truncation [Davidson, Nest Algebras]

$$\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \frac{1}{n} & \ddots \\
\ddots & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \ddots \\
\ddots & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \ddots \\
\ddots & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \ddots \\
\ddots & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \ddots \\
\ddots & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \ddots \\
\ddots & -\frac{1}{n} & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$

- the triangular truncation is unbounded on the Banach space of trace class operators
- Does there exists a trace class operator whose triangular truncation is not trace class?

Theorem [A.B.T] :

Consider the Banach Lie group $\mathrm{U}_{\mathrm{res}}(H)$, and

1. $\mathfrak{g}_+ := \mathrm{L}_{1,2}(H)/\mathfrak{u}_{1,2}(H) \subset \mathfrak{u}^*_{\mathrm{res}}(H)$,
2. $\mathbb{U} \subset T^*\,\mathrm{U}_{\mathrm{res}}(H)$, $\mathbb{U}_g = R^*_{g^{-1}}\mathfrak{g}_+$,
3. $\tilde{\pi}_r \; : \mathrm{U}_{\mathrm{res}}(H) \to \Lambda^2\mathfrak{g}_+^*$ defined by

$$\tilde{\pi}_r(g)([x_1]_{\mathfrak{u}_{1,2}}, [x_2]_{\mathfrak{u}_{1,2}}) = \Im\mathrm{Tr}(g^{-1}\, p_{\mathfrak{b}_2^+}(x_1)\, g)\left[p_{\mathfrak{u}_2}(g^{-1}\, p_{\mathfrak{b}_2^+}(x_2)\, g)\right],$$

4. $\tilde{\pi}(g) = R_g^{**}\tilde{\pi}_r(g)$.

Then $(\mathrm{U}_{\mathbf{res}}(H), \mathbb{U}, \tilde{\pi})$ **is a Banach Poisson-Lie group.**

Poisson bracket not given by a Poisson tensor

$\mathscr{H}$ separable Hilbert space

Kinetic tangent vector $X \in T_x\mathscr{H}$ equivalence classes of curves $c(t)$, $c(0) = x$, where $c_1 \sim c_2$ if they have the same derivative at 0 in a chart.

Operational tangent vector $x \in \mathscr{H}$ is a linear map $D : C_x^\infty(\mathscr{H}) \to \mathbb{R}$ satisfying Leibniz rule :

$$D(fg)(x) = Df\ g(x) + f(x)\ Dg$$

Poisson bracket not given by a Poisson tensor

Ingredients :

- Riesz Theorem
- Hahn-Banach Theorem
- compact operators $\mathscr{K}(\mathscr{H}) \subsetneq \mathscr{B}(\mathscr{H})$ bounded operators

  $\Rightarrow \exists \ell \in \mathscr{B}(\mathscr{H})^*$ such that $\ell(\mathrm{id}) = 1$ and $\ell_{|\mathscr{K}(\mathscr{H})} = 0$.

Queer tangent vector [Kriegl-Michor]

Define $D_x : C^\infty_x(\mathscr{H}) \to \mathbb{R}$, $D_x(f) = \ell(d^2(f)(x))$, where the bilinear map $d^2(f)(x)$ is identified with an operator $A \in \mathscr{B}(\mathscr{H})$ by Riesz Theorem

$$d^2(f)(x)(X, Y) = \langle X, AY \rangle$$

Then D_x is an operational tangent vector at $x \in \mathscr{H}$ of order 2

Poisson bracket not given by a Poisson tensor

Queer tangent vector [Kriegl-Michor]

$$d(fg)(x) = df(x).g(x) + f(x).dg(x)$$

$$\begin{aligned} d^2(fg)(x) &= d^2f(x).g(x) + df(x) \otimes dg(x) \\ &+ dg(x) \otimes df(x)) + f(x)d^2g(x) \end{aligned}$$

$$\begin{aligned} D_x(fg) &= \ell(d^2(fg)(x)) \\ &= \ell(d^2f(x)).g(x) + f(x)\ell(d^2g(x)) \\ &\quad + \ell(df(x) \otimes dg(x)) + \ell(dg(x) \otimes df(x)) \\ &= D_xf \; g(x) + f(x) \; D_xg \end{aligned}$$

Poisson bracket not given by a Poisson tensor

Theorem (D. Beltita, T. Golinski, A.B.Tumpach)

Consider $\mathscr{M} = \mathscr{H} \times \mathbb{R}$. Denote points of $\mathscr{M}$ as (x, λ). Then $\{\cdot, \cdot\}$ defined by

$$\{f, g\}(x, \lambda) := D_x\left(f(\cdot, \lambda)\right) \frac{\partial g}{\partial \lambda}(x, \lambda) - \frac{\partial f}{\partial \lambda}(x, \lambda) D_x\left(g(\cdot, \lambda)\right)$$

a queer Poisson bracket on $\mathscr{H} \times \mathbb{R}$, in particular it can not be represented by a bivector field $\Pi\ : T^*\mathscr{M} \times T^*\mathscr{M} \to \mathbb{R}$. The Hamiltonian vector field associated to $h(x, \lambda) = -\lambda$ is the queer operational vector field

$$X_h = \{h, \cdot\} = D_x$$

acting on $f \in C^\infty_x(\mathscr{H})$ by $D_x(f) = \ell(d^2(f)(x))$.

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Korteweg-de Vries in Lax form

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} \qquad \Leftrightarrow \qquad \frac{\partial L}{\partial t} = [(L^{\frac{3}{2}})_+, L]$$

where

- $L = D^2 + u \qquad \text{with} \qquad D = \frac{\partial}{\partial x}$
- For any pseudo-differential operator with Laurent series
  $R = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0 + a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots,$
  $R_+ = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0.$
- $L^{\frac{1}{2}} = D + a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots$ such that $(L^{\frac{1}{2}})^2 = L$

The n-th KdV hierarchy is the following hierarchy of equations indexed by $k \in \mathbb{N}$

$$\frac{\partial L}{\partial t} = [(L^{\frac{k}{n}})_+, L]$$

The restricted Grassmannian

$H = L^2(\mathbb{S}^1, \mathbb{C})$
$H = H_+ \oplus H_-$
$H_+ = \{f \in H, f(z) = a_0 + a_1 z + a_2 z^2 + \dots\}$ where $z = e^{i\theta}$
$H_- = \{f \in H, f(z) = a_{-1} z^{-1} + a_{-2} z^{-2} + a_{-3} z^{-3} + \dots\}$

$B \in GL(H_\pm, H_\pm)$ is Hilbert-Schmidt iff $\mathrm{Tr} B^* B < +\infty$

The restricted Grassmannian Gr_{res} **:** A closed subspace W of H belongs to the restricted Grassmannian Gr_{res} iff

1. $p_- \; : W \to H_-$ is Hilbert-Schmidt,
2. $p_+ \; : W \to H_+$ is Fredholm

The restricted Grassmannian

$$GL_{res} = \{\left\{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right\} \in GL(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$P_{res} = \{\left\{\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right\} \in GL(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{res} = GL_{res}/P_{res}$$

$$U_{res} = \{\left\{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right\} \in U(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{res} = U_{res}/(U(H_+) \times U(H_-))$$

Triangular group $B^+_{res} \subset GL_{res}$:

An invertible operator $g \in GL_{res}$ belongs to B^+_{res} if it is upper triangular with respect to the basis $\{z^{-n}, \dots, z^{-1}, 1, z, z^2, \dots\}$ of H, with strictly positive coefficents on the diagonal.

Remark : B^+_{res} acts on Gr_{res}

Relation between the restricted Grassmannian and the KdV hierarchy [G. Segal and G. Wilson, 1985]

$\Gamma_+ = \{g = e^f, f \ \text{ holomorphic in } \ \mathbb{D}, f(0) = 0\}$
$\Rightarrow g = e^{t_1 z + t_2 z^2 + t_3 z^3 + \cdots} \in \Gamma_+$ acts on $L^2(\mathbb{S}^1, \mathbb{C})$ by multiplication and the corresponding operator is a Toeplitz upper triangular operator in B^+_{res}.

$\mathrm{Gr}^{(n)} = \{W \in \mathrm{Gr}^0_{\mathrm{res}}(H) \ : z^n W \subset W\}$.

Proposition 5.13 in [SW85] : The action of Γ_+ on $\mathrm{Gr}^{(n)}$ induces the flows of the KdV hierarchy. For $r \geq 1$, the flow $W \mapsto \exp(t_r z^r) W$ on $\mathrm{Gr}^{(n)}$ induces the r-th KdV flow.

Key Observation : $\Gamma_+ \subset \mathrm{B}^+_{\mathrm{res}}(H)$.
Key Difficulty :
$\mathrm{B}^+_{\mathrm{res}}(H)$ **is modelled on a non-reflexive Banach space.**

Relation between the restricted Grassmannian and the KdV hierachy [G. Segal and G. Wilson, 1985]

$$\Gamma_+ = \{g = e^f, f \ \text{ holomorphic in } \ \mathbb{D}, f(0) = 0\}$$

$$\mathrm{Gr}^{(n)} = \{w \in \mathrm{Gr}^0_{\mathrm{res}}(H) \ : z^n W \subset W\}.$$

$$\Gamma_W^+ = \{g \in \Gamma_+ \ : g^{-1} W \cap H_- = \{0\}\}.$$

Proposition 5.1 in [SW85] :
$\forall W \in \mathrm{Gr}^0_{\mathrm{res}}(H)$, $\exists! \Phi_W(g, z)$ called the Baker function of W, defined for $g \in \Gamma_W^+$ and $z \in \mathbb{S}^1$, such that

(i) $\Phi_W(g, \cdot) \in W$ for each fixed $g \in \Gamma_W^+$

(ii) $\Phi_W(g, z) = g(z)(1 + \sum_1^\infty a_i(g) z^{-i})$, a_i are analytic functions on Γ_W^+ and extend to meromorphic functions on the whole of Γ^+.

Relation between the restricted Grassmannian and the KdV hierachy [G. Segal and G. Wilson, 1985]

Proposition 5.5 in [SW85] :
Set $D = \frac{\partial}{\partial x}$. For each integer $r \geq 2$, there is a unique differential operator P_r of the form $P_r = D^r + p_{r2}D^{r-2} + \cdots + p_{r,r-1}D + p_{rr}$ such that $\frac{\partial \Phi_W}{\partial t_r} = P_r \Phi_W$.

Denote by $\mathscr{C}^{(n)}$ the space of all operators P_n associated to subspaces W in $\mathrm{Gr}^{(n)}$ evaluated at $t_2 = t_3 = \cdots = 0$.

Proposition 5.13 in [SW85] : The action of Γ_+ on $\mathrm{Gr}^{(n)}$ induces an action on the space $\mathscr{C}^{(n)}$. For $r \geq 1$, the flow $W \mapsto \exp(t_r z^r)W$ on $\mathrm{Gr}^{(n)}$ induces the r-th KdV flow on $\mathscr{C}^{(n)}$.

Key Observation : $\Gamma_+ \subset \mathrm{B}^+_{\mathbf{res}}(H)$.

Poisson geometry of the restricted Grassmannian and dressing action leading to the KdV equation

Theorem [A.B.T] : The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}(H) = \mathrm{U}_{\mathrm{res}}(H)/\,\mathrm{U}(H_+) \times \mathrm{U}(H_-) = \mathrm{GL}_{\mathrm{res}}(H)/\,\mathrm{P}_{\mathrm{res}}(H)$ carries a natural Poisson structure such that :

1. the projection $p\ : \mathrm{U}_{\mathrm{res}}(H) \to \mathrm{Gr}_{\mathrm{res}}(H)$ is a Poisson map,
2. the natural action of $\mathrm{U}_{\mathrm{res}}(H)$ on $\mathrm{Gr}_{\mathrm{res}}(H)$ is a Poisson map,
3. the following right action of $\mathrm{B}^+_{\mathrm{res}}(H)$ on $\mathrm{Gr}_{\mathrm{res}}(H)$ is a Poisson map :

$$\begin{array}{ccc} \mathrm{Gr}_{\mathrm{res}}(H) \times \mathrm{B}^+_{\mathrm{res}}(H) & \to & \mathrm{Gr}_{\mathrm{res}}(H) \\ (g\,\mathrm{P}_{\mathrm{res}}(H), b) & \mapsto & (b^{-1}g)\,\mathrm{P}_{\mathrm{res}}(H). \end{array}$$

4. the symplectic leaves of $\mathrm{Gr}_{\mathrm{res}}(H)$ are the Schubert-Bruhat cells and are the orbits of $\mathrm{B}^+_{\mathrm{res}}(H)$.

$\Rightarrow$ the action of $\Gamma^+ \subset \mathrm{B}^+_{\mathrm{res}}(H)$ is Poisson and generates the KdV hierarchy.