Riemann surfaces of class O_G, the Fenchel-Nielsen coordinates and holomorphic quadratic differentials

Teichmüller space: from low dimension to infinity and beyond
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Infinite Riemann surfaces

- A Riemann surface $X = \mathbb{H}/\Gamma$ is infinite if Γ is infinitely generated.
- Γ is of the first kind if $\Lambda(\Gamma) = \partial_\infty \mathbb{H}$; otherwise Γ is of the second kind
- If Γ is of the second kind then $\partial_\infty \mathbb{H}/\Gamma$ contains a closed curve or an open arc.

Definition

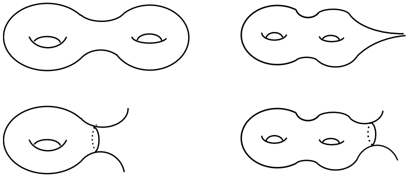
The geodesic flow is a map $g : \mathbb{R} \times T^1X \to T^1X$ which moves $\mathbf{v} \in T^1X$ by unit speed along a geodesic tangent to $\mathbf{v}$.

- The Liouville measure L on the unit tangent bundle T^1X is locally the product of of the hyperbolic area and the angle measure
- L is invariant under the geodesic flow
- The geodesic flow on X is ergodic if for every invariant set $A \subset T^1X$ either $L_X(A) = 0$ or $L_X(A^c) = 0$.

Finite Riemann surfaces

If $\pi_1(S)$ is finitely generated then

$$\text{the geodesic flow is ergodic} \Leftrightarrow \textit{area}(S) < \infty$$



All infinite Riemann surfaces have infinite area.

Characterizations of ergodicity of the geodesic flow

Theorem (Hopf-Tsuji-Sullivan, Astala-Zinsmeister-Bishop)

Let $X = \mathbb{H}/\Gamma$ be an infinite Riemann surface. The following are equivalent:

1. *The geodesic flow on X is ergodic.*
2. *The Poincaré series diverges, i.e., $\sum_{\gamma \in \Gamma} e^{-d_{hyp}(z,\gamma(z))} = \infty$.*
3. *Brownian motion on X is recurrent.*
4. *X satisfies the Bowen property*

Theorem (Ahlfors-Sario, Poincaré)

The following are equivalent to the above:

1. *X does not support a Green's function, i.e. $X \in O_G$.*
2. *The harmonic measure of $\partial_\infty X$ is zero.*
3. *The extremal distance between a compact subsurface of X and $\partial_\infty X$ is infinite.*

Riemann surfaces of class O_G

Classification Problem. Determine if an explicitly given Riemann surface $X \in O_G$ (or geodesic flow of X is ergodic).

- (Ahlfors-Sario) X planar; $X = \mathbb{C} \setminus E \in O_G \Leftrightarrow \mathrm{Cap}(E) = 0$.
- (Tsuji, Laasonen) If $X = \mathbb{D}/\Gamma$ and D is a Dirichlet fundamental polygon of Γ, then

$$\sigma(D \cap \mathbb{D}_r) \geq \frac{c}{1-r} \qquad \Rightarrow \quad X \notin O_g$$

$$\sigma(D \cap \mathbb{D}_r) \leq c \log \frac{1}{1-r} \Rightarrow \quad X \in O_G,$$

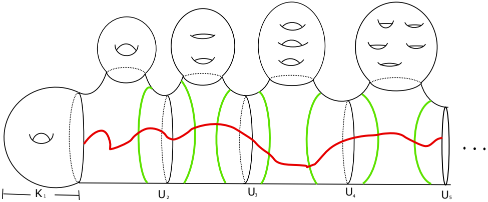
  where $\mathbb{D}_r = \{|z| < r\}$, $\sigma(\cdot)$ - hyperbolic area in $\mathbb{D}$.
- (Nicholls) There is no characterization of class O_G in terms of the growth rate of $\sigma(D \cap \mathbb{D}_r)$.
- (Nicholls) $\exists \Gamma_1, \Gamma_2 < PSL_2(\mathbb{R})$ with a common fundamental infinite convex polygon P such that $\mathbb{D}/\Gamma_1 \in O_G$, but $\mathbb{D}/\Gamma_2 \notin O_G$.
- (Fernández-Rodrigues) $\exists$ a Riemann surface X and $\alpha_0, t_0 > 0$ s.t. for all $t > t_0$

$$\sigma_X(B(p,t)) \geq e^{\alpha_0 t},$$

  but $X \in O_G$. Here $B(p,t)$ is the metric ball of radius t, centered at $p \in X$.

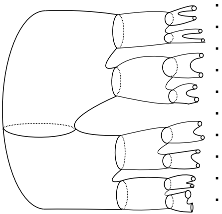
- A topological end of an infinite Riemann surface S is a "way of going to infinity" in S.

  Example - Loch Ness monster: single topological end and infinite genus

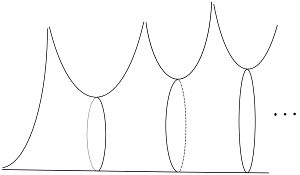


Examples

Cantor Tree: (add countably many handles) Cantor set of ends



Infinite Flute Surface: space of ends is $\mathbb{N} \cup \infty$



Pairs of pants decomposition

- A pair of pants is a topological space homeomorphic to the 2-sphere minus three topological disk.
- A geodesic pair of pants is a pair of pants equipped with the hyperbolic metric such that its three boundary curves are geodesics.

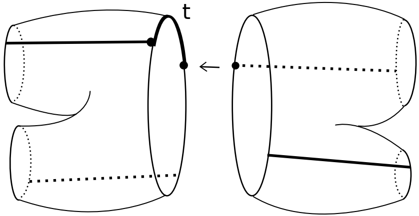
Theorem (Alvarez-Rodriguez, Basmajian-Š.)

If Γ is of the first kind then any topological pants decomposition of $X = \mathbb{H}/\Gamma$ can be straightened to a geodesic pants decomposition.

- If Γ is not of the first kind then X is not parabolic since X contains a geodesic half-plane.

Gluing two geodesic pairs of pants

- Two geodesic pairs of pants with two cuffs of equal length can be glued by an isometry; the choice of gluing is given by the relative twists: $-1/2 \leq t \leq 1/2$



- Fenchel-Nielsen parameters of X is the collection $\{(l_X(\alpha_n), t_X(\alpha_n))\}_n$.
- Conversely, given a topological pants decomposition of S and a collection $\{(l(\alpha_n), t(\alpha_n))\}$, there exists a hyperbolic surface X with these Fenchel-Nielsen parameters.
- However, the surface X obtained from $\{(l(\alpha_n), t(\alpha_n))\}$ might be incomplete-i.e., $\Lambda(\Gamma_X) \neq \partial_\infty \mathbb{H}$.

The Fenchel-Nielsen coordinates and groups of the first kind

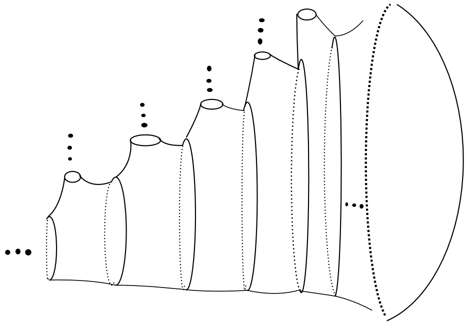


Figure: An incomplete surface

Theorem (Basmajian-Š.)

Let X be obtained from the Fenchel-Nielsen coordinates $\{(l(\alpha_n), t(\alpha_n))\}_n$. If X is incomplete, then we can change the twists $t(\alpha_n)$ and keep the original lengths such that the new hyperbolic surface is complete.

The class O_G from cuff lengths (Loch Ness Monster)

- If $X = \mathbb{H}/\Gamma \in O_G$ then G is of the first kind-i.e., it is union of geodesic pairs of pants.
- However, the first kind does not imply parabolicity.

Theorem (Basmajian, Hakobyan and Š.)

Let X be an infinite Loch Ness monster with cuffs of length $\{l_n\}_{n\in\mathbb{N}}$ accumulating to the topological end of X. Let $\{f_n\}_{n\in\mathbb{N}}$ be the lengths of geodesics that cut off the genus. If

$$f_n \leq M < \infty, \forall n \in \mathbb{N},$$

and

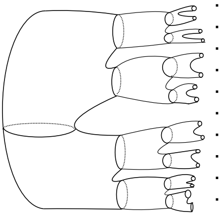
$$\sum_{n=1}^{\infty} e^{-\frac{l_n}{2}} = \infty.$$

then $X \in O_G$.

- In particular, if X is an infinite flute surface then $\sum_{n=1}^{\infty} e^{-\frac{l_n}{2}} = \infty$ implies that $X \in O_G$ (no matter what the twists are).
- $\ell_n = 2\log n$ satisfies the above condition.

The class O_G from cuff lengths (Blooming Cantor Tree)

Cantor Tree: (add countably many handles) Cantor set of ends



Theorem (Basmajian, Hakobyan and Š.)

Let X be a Blooming Cantor Tree surface with a Cantor set of ends. If for every cuff α of generation n,

$$l_X(\alpha) \lesssim \frac{n}{2^n},$$

then $X \in O_G$.

- (McMullen) If $C \geq l_X(\alpha) \geq 1/C > 0$ then $X \notin O_G$ (yet it is complete).
- We obtain a general sufficient condition which works for a Riemann surface with arbitrary topology, and which implies all the results above.
- This general condition is formulated in terms of the moduli of certain annuli embedded in the surface.

The modulus of curve family

- Let Γ be a curve family of locally rectifiable curves on a Riemann surface X
- An allowable metric for Γ is a Borel measurable differential $\rho(z)|dz|$ on X s.t.

$$\int_\gamma \rho(z)|dz| \geq 1, \quad \forall \gamma \in \Gamma$$

.

- The modulus of Γ is defined by

$$\mathrm{mod}\Gamma = \inf \iint_X \rho(z)^2 dxdy,$$

where the infimum is over all Γ allowable differentials.

- Let $D \subset X$ be open and $E_1, E_2 \subset \bar{D}$ two closed subsets.
Extremal distance between E_1 and E_2 in $\bar{D}$ is

$$\lambda_D(E_1, E_2) = \frac{1}{\mathrm{mod}\Gamma}$$

where Γ is the family of curves in D connecting E_1 and E_2.

The extremal distance condition for $X \in O_G$

- Let $\{K_n\}_{n=1}^{\infty}$ be a compact exhaustion of a Riemann surface X by regular subsurfaces whose boundary components are not null homotopic.
- (Ahlfors-Sario) X is parabolic $\iff \lambda_{K_n \setminus K_1}(\partial K_1, \partial K_n) \underset{n \to \infty}{\longrightarrow} \infty$.
- Suppose $\partial K_n \subset U_n \subset K_{n+1} \setminus K_{n-1}$ with $\partial U_n = a_n \cup b_n$, $a_n \subset K_n^{\circ}$ and $b_n \subset (K_{n+1} \setminus K_n)^{\circ}$. Let λ_n is the extremal distance between a_n and b_n in U_n.
- By the serial rule for extremal distance: If $\sum_{n=1}^{\infty} \lambda_n = \infty$ then $X \in O_G$.

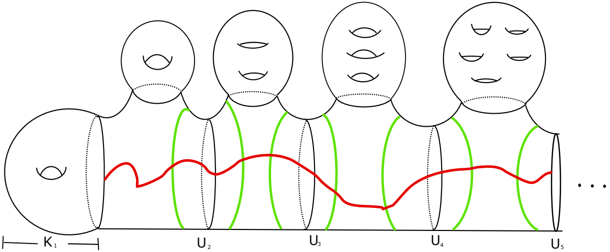
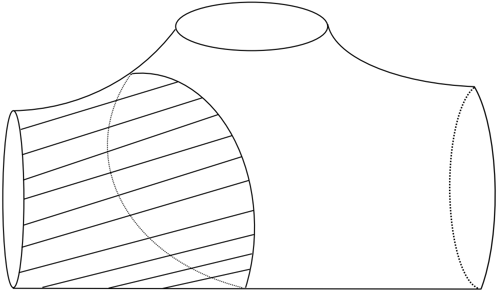


Figure: The serial rule

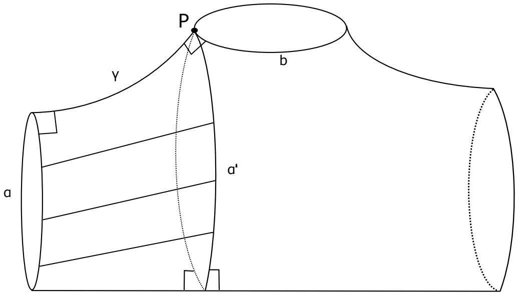
Collars on hyperbolic surfaces

- The standard one-sided collar of a simple closed geodesic α is the set of all points on one side of α which are at most $r(\ell/2) := \sinh^{-1} \frac{1}{\sinh(\ell/2)}$ away from α.
- (Maskit) Let $R_{st}(\alpha)$ be a one-sided standard collar of α. The extremal distance $\lambda_{R_{st}(\alpha)}$ between boundary curves of R satisfies

$$\lambda_{R_{st}(\alpha)} = \frac{e^{-\frac{\ell}{2}}}{\ell}.$$



The non-standard one-sided collars



- The non-standard collar $R_{ns}(\alpha)$ around geodesic α of length ℓ.
- Let γ be the orthogeodesic between α and β, let $\lambda_{R_{ns}(\alpha)}$ be the extremal distance between the boundary curves of $R_{ns}(\alpha)$
- (Basmajian, Hakobyan and Š.) When $\ell \to \infty$, we have

$$\lambda_{R_{ns}(\alpha)} \asymp \gamma e^{-\ell/2}$$

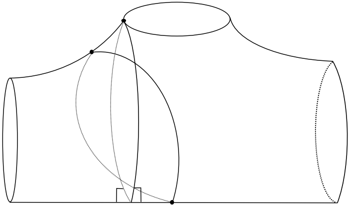
for all $0 < \gamma < \gamma_0$ (or equivalently β large) and

$$\lambda_{R_{ns}(\alpha)} \asymp e^{-\ell/2}$$

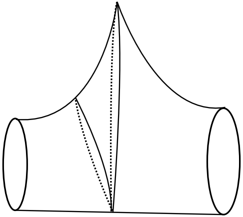
for all $\gamma \geq \gamma_0$ (or β bounded from above).

Comparing the non-standard and standard collars

In general, the non-standard and standard collars do not contain each other.

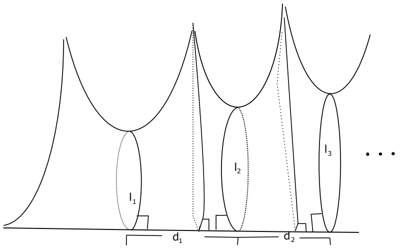


When one cuffs is a puncture-i.e., $\ell(\beta) = 0$ then



Note that $\frac{\lambda_{R_{ns}(\alpha)}}{\lambda_{R_{st}(\alpha)}} \asymp \ell$.

The zero twist flute surfaces



- The distance d_n between ℓ_n and ℓ_{n+1} is approximately $e^{-\ell_n/2} + e^{-\ell_{n+1}/2}$
- $X = \{(\ell_n, 0)\}$ is incomplete iff $\sum_{n=1}^{\infty} d_n < \infty$
- For zero twist flute surface $X(\ell_n, 0)$ we have a complete understanding.

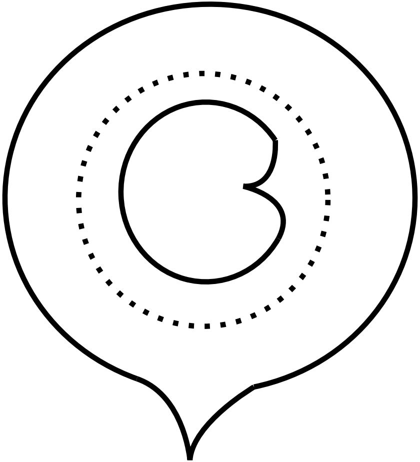
Proposition. (Basmajian-Hakobyan-Š.)

Let $X(\ell_n, 0)$ be a flute surface with zero twists. Then the following are equivalent

- $X(\ell_n, 0) \in O_G$,
- $\sum_n e^{-\ell_n/2} = \infty$,
- $X(\ell_n, 0)$ *is complete, i.e.* Γ *is of the first kind.*

Gluing non-standard collars

When we glue two one-sided non-standard collars along a common geodesic we obtain an annulus with two elongated pieces corresponding to (half of) neighborhood of punctures whose relative positions depend on the twist



Twist effects: flute surfaces

- When gluing two standard one-sided collars the twist has no effect on the shape of the obtained annulus and the extremal distance does not change with the twist.
- When gluing two non-standard collars the shape depends on the twist and the extremal distance increases with the absolute value of the twist.

Theorem (Basmajian, Hakobyan and Š.)

Let X be an infinite flute with the Fenchel-Nielsen coordinates $\{(\ell_n, t_n)\}_n$. If

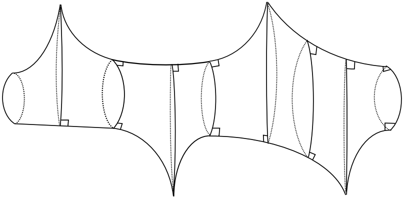
$$\sum_{n=1}^{\infty} e^{-(\frac{1}{2}-\frac{|t_n|}{2})\ell_n} = \infty$$

then X is parabolic.

- If $\sum_{n=1}^{\infty} e^{-\frac{1}{2}\ell_n} = \infty$ then $X \in O_G$ for all choices of twists t_n.
- It is possible that $\sum_{n=1}^{\infty} e^{-\frac{1}{2}\ell_n} < \infty$ and yet $X \in O_G$.

Twist effects: flute surfaces with half-twists

- let $\ell_n = 4\log n$ and $t_n \equiv 1/2$
- $\sum_n e^{-\ell_n/2} \leq \sum_n n^{-2} < \infty$
- $\sum_n e^{-(\frac{1}{2}-\frac{|t_n|}{2})\ell_n} = \sum_n e^{-\frac{\ell_n}{4}} = \sum_n 1/n = \infty$ then $X(4\log n, 1/2) \in O_G$



When $t_n \equiv 1/2$, to which extent $\sum_{n=1}^{\infty} e^{-\ell_n/4} = \infty$ characterize the class O_g?

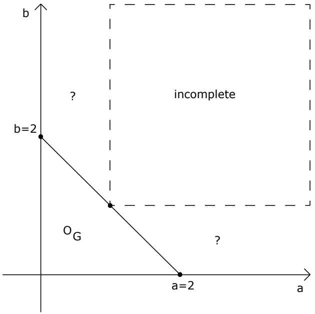
The non-complete half-twist surfaces

Theorem (Basmajian, Hakobyan and Š.)

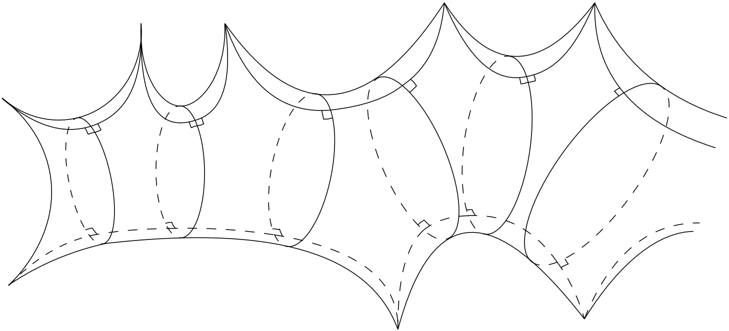
Let $X(\ell_n, 1/2)$ be a half-twist flute surface with concave *and increasing lengths ℓ_n. TFAE:*

- $X(\ell_n, 1/2) \in O_G$,
- $\sum_{n=1}^{\infty} e^{-\ell_n/4} = \infty$,
- $X(\ell_n, 1/2)$ *is complete.*

Hakobyan slice: $\ell_{2n} = a \log(n+1) + b \log n$, $\ell_{2n+1} = (a+b)\log(n+1)$, $t_n \equiv 1/2$ for $a > 0$ and $b > 0$; ℓ_n increasing but not concave



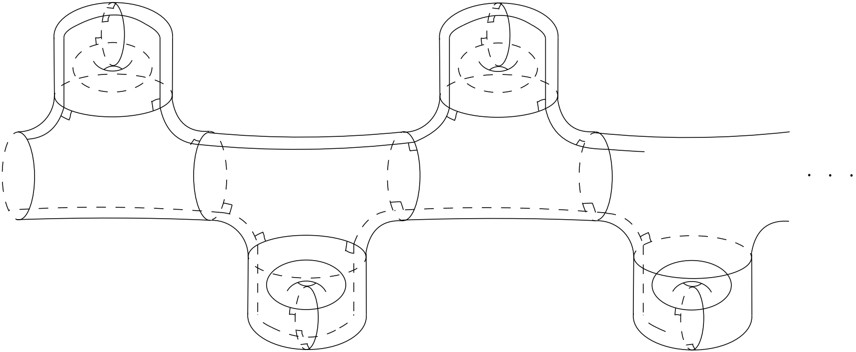
A Riemann surface X is symmetric if there is an orientation reversing isometry which interchanges front to back decomposition of pairs of pants into hexagons.



Theorem (M. Pandazis and Š.)

Let $X_f = \mathbb{H}/\Gamma$ be a flute surface with $t_n \in \{0, \frac{1}{2}\}$ for all n. Then $X_f \in O_g$ if and only if Γ is of the first kind (i.e. X_f complete-no funnels or half-planes).

More generally, a surface is end symmetric if each component of the complement of a compact geodesic subsurface has an orientation reversing isometry whose set of fixed points divides the end into front and back.



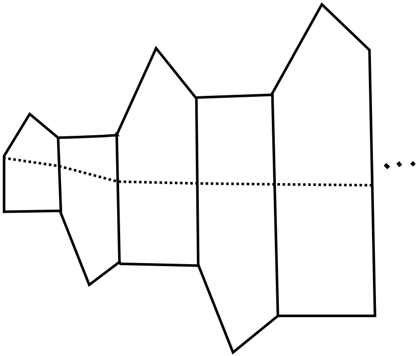
Theorem (M. Pandazis and Š.)

Let $X = \mathbb{H}/\Gamma$ be an end symmetric Riemann surface with finitely many ends. Then $X \in O_G$ iff Γ is of the first kind.

Hakobyan slice

We need to find a condition on the lengths such that the half-twist flute surface X_f is complete.

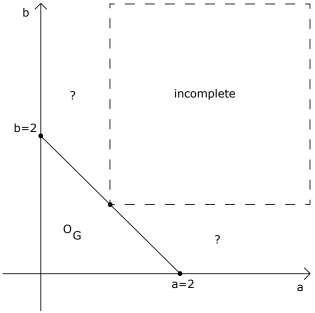
for $\sigma_n = \ell_n - \ell_{n-1} + \cdots + (-1)^{n-1}\ell_1$, the expression $e^{-\sigma_n/2}$ is the length of the summit of a Sacherri quadrilateral between ℓ_n and ℓ_{n+1} obtained by concatenations



Proposition. (Basmajian-Hakobyan-Š.)

Let $X = \mathbb{H}/\Gamma$ be a half-flute with cuff lengths $\{\ell_n\}$. If $\sum_{n=1}^{\infty} e^{-\sigma_n/2} < \infty$ then Γ is of the second kind (i.e. X incomplete).

In the ?-regions of Hakobyan slice, we have $\sum_{n=1}^{\infty} e^{-\sigma_n/2} = \infty$.

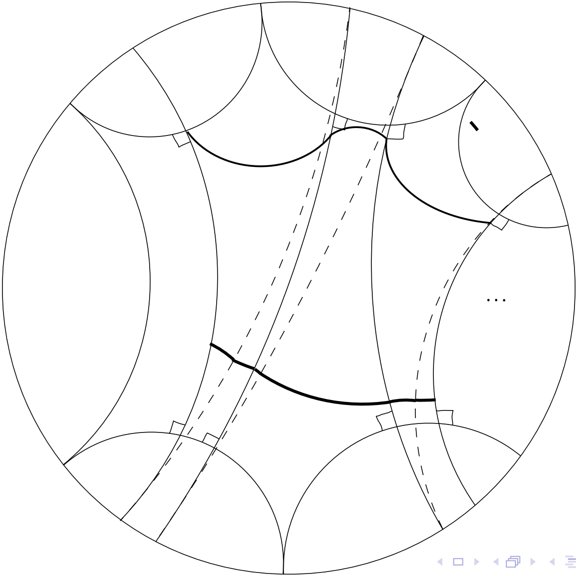


Theorem (Pandazis-Š.)

Let X be a half-twist flute surface in the Hakobyan slice. Then $\sum_{n=1}^{\infty} e^{-\sigma_n/2} = \infty$ implies $X \in O_G$.

Hakobyan slice

We compare $\sum_{n=1}^{\infty} e^{-\sigma_n/2} = \infty$ to the length of a piecewise horocyclic path that follows a zig-zag of geodesics obtained by adding a geodesic asymptotic to ℓ_n and ℓ_{n+1} at its ends.



Integrable quadratic differentials

Analogue of Hubbard-Masur theorem for compact surfaces.

Theorem (Š.)

Let $X = \mathbb{H}/\Gamma$ be an infinite Riemann surface with Γ of the first kind. Then the space of integrable holomorphic quadratic differentials $A(X)$ on X is homeomorphic to a subset $ML_f(X)$ of the space of all measured laminations on X, where $ML_f(X)$ can be realized by partial foliations on X with finite Dirichlet energy.

Theorem (Š.)

$X \notin O_G$ iff there exists $\varphi \in A(X)$ whose horizontal trajectories are escaping to $\partial_\infty X$.

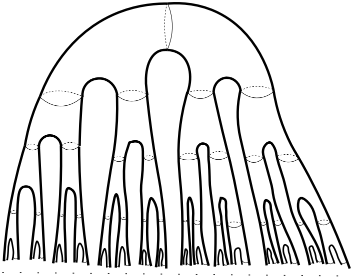
Brownian motion on X is recurrent iff a.e. leaf of every finite-area holomorphic quadratic differential is recurrent.

Theorem (Š.)

Assume $X \in O_G$. The Teichmüller distance is given by Kerkchoff's formula, i.e.

$$d_T(Y, Z) = \frac{1}{2} \log \sup_{\gamma \in S} \frac{\mathrm{ext}_Z(\gamma)}{\mathrm{ext}_Y(\gamma)}$$

where $\mathrm{ext}_Y(\cdot)$ and $\mathrm{ext}_Z(\cdot)$ are extremal lengths on surfaces Y and Z of the corresponding simple closed curves.



Theorem (Š.)

Let X_C be the Cantor tree surface with geodesic pants decomposition such that the lengths of the boundary geodesics at the level n are equal to some ℓ_n for each n. If there exists $r > 2$ such that

$$\ell_n = \frac{n^r}{2^n}$$

then $X \notin O_G$.

Bridges the gap between Basmajian-Hakobyan-Š. ($\ell_n \leq n/2^n$ implies $X \in O_G$) and McMullen ($\ell_n \geq c_0 > 0$ implies $X \notin O_G$).

Cantor tree surfaces

The proof is by constructing a finite-area holomorphic quadratic differential on X whose horizontal trajectories are escaping ot $\partial_\infty X$.

Theorem (Pandazis)

Let X_C be the (blooming) Cantor tree surface with geodesic pants decomposition such that there exists $r > 1$ with

$$\frac{1}{n^2} \lesssim \ell_n^j \lesssim \frac{n^r}{2^n},$$

for all $1 \leq j \leq 2^{n+1}$ then $X \notin O_G$.

The idea is to construct a partial foliation of X_C using the pants decomposition and breaking each pair of pants into hexagons.

Problem 1. (Sullivan) A flute surface is conformal to either a complex plane minus a discrete set of points ($\in O_G$) or a disk minus a discrete set of points ($\notin O_G$). If the points accumulate to the whole S^1 then the surface has covering group of the first kind but it is not in O_G. Find the Fenchel-Nielsen coordinates of such flute surfaces.

Problem 2. (Kahn, Markovic) Given any sequence $\{a_n\}_n$ of positive numbers, is there a flute surface X with cuff lengths $\ell_n = a_n$ and a choice of twists $\{t_n\}_n$ such that $X \in O_G$? Basmajian-Š. proved that there is always a choice of twists $\{t_n\}$ such that the covering group of X is of the first kind, i.e. X is complete.

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Thank you!