Riemann surfaces of class $O_G$, the Fenchel-Nielsen coordinates and holomorphic quadratic differentials

Teichmüller space: from low dimension to infinity and beyond
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A Riemann surface $X = \mathbb{H}/\Gamma$ is infinite if $\Gamma$ is infinitely generated.

$\Gamma$ is of the first kind if $\Lambda(\Gamma) = \partial_{\infty}\mathbb{H}$; otherwise $\Gamma$ is of the second kind.

If $\Gamma$ is of the second kind then $\partial_{\infty}\mathbb{H}/\Gamma$ contains a closed curve or an open arc.

**Definition**

The **geodesic flow** is a map $g : \mathbb{R} \times T^1X \to T^1X$ which moves $v \in T^1X$ by unit speed along a geodesic tangent to $v$.

- The **Liouville measure** $L$ on the unit tangent bundle $T^1X$ is locally the product of of the hyperbolic area and the angle measure.
- $L$ is invariant under the geodesic flow.
- The geodesic flow on $X$ is **ergodic** if for every invariant set $A \subset T^1X$ either $L_X(A) = 0$ or $L_X(A^c) = 0$. 
Finite Riemann surfaces

If $\pi_1(S)$ is finitely generated then

the geodesic flow is ergodic $\iff \text{area}(S) < \infty$

All infinite Riemann surfaces have infinite area.
Characterizations of ergodicity of the geodesic flow

**Theorem (Hopf-Tsuji-Sullivan, Astala-Zinsmeister-Bishop)**

Let $X = \mathbb{H}/\Gamma$ be an infinite Riemann surface. The following are equivalent:

1. The geodesic flow on $X$ is ergodic.
2. The Poincaré series diverges, i.e., $\sum_{\gamma \in \Gamma} e^{-d_{hyp}(z, \gamma(z))} = \infty$.
3. Brownian motion on $X$ is recurrent.
4. $X$ satisfies the Bowen property.

**Theorem (Ahlfors-Sario, Poincaré)**

The following are equivalent to the above:

1. $X$ does not support a Green’s function, i.e. $X \in O_G$.
2. The harmonic measure of $\partial_\infty X$ is zero.
3. The extremal distance between a compact subsurface of $X$ and $\partial_\infty X$ is infinite.
Classification Problem. Determine if an explicitly given Riemann surface \( X \in O_G \) (or geodesic flow of \( X \) is ergodic).

- (Ahlfors-Sario) \( X \) planar; \( X = \mathbb{C} \setminus E \in O_G \leftrightarrow \text{Cap}(E) = 0 \).
- (Tsuji, Laasonen) If \( X = \mathbb{D}/\Gamma \) and \( D \) is a Dirichlet fundamental polygon of \( \Gamma \), then
  \[
  \sigma(D \cap \mathbb{D}_r) \geq \frac{c}{1 - r} \quad \Rightarrow \quad X \notin O_g
  \]
  \[
  \sigma(D \cap \mathbb{D}_r) \leq c \log \frac{1}{1 - r} \quad \Rightarrow \quad X \in O_G,
  \]
  where \( \mathbb{D}_r = \{|z| < r\} \), \( \sigma(\cdot) \) - hyperbolic area in \( \mathbb{D} \).
- (Nicholls) There is no characterization of class \( O_G \) in terms of the growth rate of \( \sigma(D \cap \mathbb{D}_r) \).
- (Nicholls) \( \exists \Gamma_1, \Gamma_2 < PSL_2(\mathbb{R}) \) with a common fundamental infinite convex polygon \( P \) such that \( \mathbb{D}/\Gamma_1 \in O_G \), but \( \mathbb{D}/\Gamma_2 \notin O_G \).
- (Fernández-Rodrigues) \( \exists \) a Riemann surface \( X \) and \( \alpha_0, t_0 > 0 \) s.t. for all \( t > t_0 \)
  \[
  \sigma_X(B(p, t)) \geq e^{\alpha_0 t},
  \]
  but \( X \in O_G \). Here \( B(p, t) \) is the metric ball of radius \( t \), centered at \( p \in X \).
Examples of infinite Riemann surfaces

- A **topological end** of an infinite Riemann surface $S$ is a “way of going to infinity” in $S$.

**Example** - Loch Ness monster: single topological end and infinite genus
Examples

**Cantor Tree**: (add countably many handles) Cantor set of ends

**Infinite Flute Surface**: space of ends is $\mathbb{N} \cup \infty$
A **pair of pants** is a topological space homeomorphic to the 2-sphere minus three topological disk.

A **geodesic pair of pants** is a pair of pants equipped with the hyperbolic metric such that its three boundary curves are geodesics.

**Theorem (Alvarez-Rodriguez, Basmajian-Š.)**

If $\Gamma$ is of the first kind then any topological pants decomposition of $X = \mathbb{H}/\Gamma$ can be straightened to a geodesic pants decomposition.

If $\Gamma$ is not of the first kind then $X$ is not parabolic since $X$ contains a geodesic half-plane.
Gluing two geodesic pairs of pants

- Two geodesic pairs of pants with two cuffs of equal length can be glued by an isometry; the choice of gluing is given by the relative twists: $-1/2 \leq t \leq 1/2$

![Diagram of two geodesic pairs of pants being glued](image)

- **Fenchel-Nielsen parameters** of $X$ is the collection $\{(l_X(\alpha_n), t_X(\alpha_n))\}_n$.

- Conversely, given a topological pants decomposition of $S$ and a collection $\{(l(\alpha_n), t(\alpha_n))\}$, there exists a hyperbolic surface $X$ with these Fenchel-Nielsen parameters.

- However, the surface $X$ obtained from $\{(l(\alpha_n), t(\alpha_n))\}$ might be incomplete—i.e., $\Lambda(\Gamma_X) \neq \partial_\infty \mathbb{H}$. 
Theorem (Basmajian-Š.)

Let $X$ be obtained from the Fenchel-Nielsen coordinates $\{(l(\alpha_n), t(\alpha_n))\}_n$. If $X$ is incomplete, then we can change the twists $t(\alpha_n)$ and keep the original lengths such that the new hyperbolic surface is complete.
The class $O_G$ from cuff lengths (Loch Ness Monster)

- If $X = \mathbb{H}/\Gamma \in O_G$ then $G$ is of the first kind-i.e., it is union of geodesic pairs of pants.
- However, the first kind does not imply parabolicity.

Theorem (Basmajian, Hakobyan and Š.)

Let $X$ be an infinite Loch Ness monster with cuffs of length $\{l_n\}_{n \in \mathbb{N}}$ accumulating to the topological end of $X$. Let $\{f_n\}_{n \in \mathbb{N}}$ be the lengths of geodesics that cut off the genus. If

$$f_n \leq M < \infty, \forall n \in \mathbb{N},$$

and

$$\sum_{n=1}^{\infty} e^{-\frac{l_n}{2}} = \infty.$$

then $X \in O_G$.

- In particular, if $X$ is an infinite flute surface then $\sum_{n=1}^{\infty} e^{-\frac{l_n}{2}} = \infty$ implies that $X \in O_G$ (no matter what the twists are).
- $\ell_n = 2 \log n$ satisfies the above condition.
The class $O_G$ from cuff lengths (Blooming Cantor Tree)

Cantor Tree: (add countably many handles) Cantor set of ends

Theorem (Basmajian, Hakobyan and Š.)

Let $X$ be a Blooming Cantor Tree surface with a Cantor set of ends. If for every cuff $\alpha$ of generation $n$,

$$l_X(\alpha) \lesssim \frac{n}{2^n},$$

then $X \in O_G$.

- (McMullen) If $C \geq l_X(\alpha) \geq 1/C > 0$ then $X \notin O_G$ (yet it is complete).
- We obtain a general sufficient condition which works for a Riemann surface with arbitrary topology, and which implies all the results above.
- This general condition is formulated in terms of the moduli of certain annuli embedded in the surface.
Let $\Gamma$ be a curve family of locally rectifiable curves on a Riemann surface $X$.

An allowable metric for $\Gamma$ is a Borel measurable differential $\rho(z)|dz|$ on $X$ s.t.

$$\int_{\gamma} \rho(z)|dz| \geq 1, \quad \forall \gamma \in \Gamma.$$ 

The modulus of $\Gamma$ is defined by

$$\text{mod} \Gamma = \inf \int\int_{\Gamma} \rho(z)^2 \, dx \, dy,$$

where the infimum is over all $\Gamma$ allowable differentials.

Let $D \subset X$ be open and $E_1, E_2 \subset \overline{D}$ two closed subsets. Extremal distance between $E_1$ and $E_2$ in $\overline{D}$ is

$$\lambda_D(E_1, E_2) = \frac{1}{\text{mod} \Gamma}$$

where $\Gamma$ is the family of curves in $D$ connecting $E_1$ and $E_2$. 
The extremal distance condition for $X \in O_G$

- Let $\{K_n\}_{n=1}^{\infty}$ be a compact exhaustion of a Riemann surface $X$ by regular subsurfaces whose boundary components are not null homotopic.

- (Ahlfors-Sario) $X$ is parabolic $\iff \lambda_{K_n \setminus K_1}(\partial K_1, \partial K_n) \xrightarrow{n \to \infty} \infty$.

- Suppose $\partial K_n \subset U_n \subset K_{n+1} \setminus K_{n-1}$ with $\partial U_n = a_n \cup b_n$, $a_n \subset K_1^\circ$ and $b_n \subset (K_{n+1} \setminus K_n)^\circ$. Let $\lambda_n$ be the extremal distance between $a_n$ and $b_n$ in $U_n$.

- By the serial rule for extremal distance: If $\sum_{n=1}^{\infty} \lambda_n = \infty$ then $X \in O_G$.

Figure: The serial rule
The standard one-sided collar of a simple closed geodesic $\alpha$ is the set of all points on one side of $\alpha$ which are at most $r(\ell/2) := \sinh^{-1} \frac{1}{\sinh(\ell/2)}$ away from $\alpha$.

(Maskit) Let $R_{st}(\alpha)$ be a one-sided standard collar of $\alpha$. The extremal distance $\lambda_{R_{st}(\alpha)}$ between boundary curves of $R$ satisfies

$$\lambda_{R_{st}(\alpha)} = \frac{e^{-\frac{\ell}{2}}}{\ell}.$$
The non-standard collar $R_{ns}(\alpha)$ around geodesic $\alpha$ of length $\ell$.

Let $\gamma$ be the orthogeodesic between $\alpha$ and $\beta$, let $\lambda_{R_{ns}(\alpha)}$ be the extremal distance between the boundary curves of $R_{ns}(\alpha)$

(Basmajian, Hakobyan and Š.) When $\ell \to \infty$, we have

$$\lambda_{R_{ns}(\alpha)} \preceq \gamma e^{-\ell/2}$$

for all $0 < \gamma < \gamma_0$ (or equivalently $\beta$ large) and

$$\lambda_{R_{ns}(\alpha)} \preceq e^{-\ell/2}$$

for all $\gamma \geq \gamma_0$ (or $\beta$ bounded from above).
Comparing the non-standard and standard collars

In general, the non-standard and standard collars do not contain each other.

When one cuffs is a puncture—i.e., $\ell(\beta) = 0$ then

Note that $\frac{\lambda_{Rns}(\alpha)}{\lambda_{Rst}(\alpha)} \asymp \ell$. 
The distance $d_n$ between $\ell_n$ and $\ell_{n+1}$ is approximately $e^{-\ell_n/2} + e^{-\ell_{n+1}/2}$

$X = \{(\ell_n, 0)\}$ is incomplete iff $\sum_{n=1}^{\infty} d_n < \infty$

For zero twist flute surface $X(\ell_n, 0)$ we have a complete understanding.

**Proposition. (Basmajian-Hakobyan-Š.)**

Let $X(\ell_n, 0)$ be a flute surface with zero twists. Then the following are equivalent

- $X(\ell_n, 0) \in O_G$,
- $\sum_n e^{-\ell_n/2} = \infty$,
- $X(\ell_n, 0)$ is complete, i.e. $\Gamma$ is of the first kind.
When we glue two one-sided non-standard collars along a common geodesic we obtain an annulus with two elongated pieces corresponding to (half of) neighborhood of punctures whose relative positions depend on the twist.
Twist effects: flute surfaces

- When gluing two standard one-sided collars the twist has no effect on the shape of the obtained annulus and the extremal distance does not change with the twist.
- When gluing two non-standard collars the shape depends on the twist and the extremal distance increases with the absolute value of the twist.

Theorem (Basmajian, Hakobyan and Š.)

Let $X$ be an infinite flute with the Fenchel-Nielsen coordinates $\{(\ell_n, t_n)\}_n$. If

$$\sum_{n=1}^{\infty} e^{-\left(\frac{1}{2} - \frac{|t_n|}{2}\right)\ell_n} = \infty$$

then $X$ is parabolic.

- If $\sum_{n=1}^{\infty} e^{-\frac{1}{2}\ell_n} = \infty$ then $X \in O_G$ for all choices of twists $t_n$.
- It is possible that $\sum_{n=1}^{\infty} e^{-\frac{1}{2}\ell_n} < \infty$ and yet $X \in O_G$. 
Twist effects: flute surfaces with half-twists

- Let $\ell_n = 4 \log n$ and $t_n \equiv 1/2$
- $\sum_n e^{-\ell_n/2} \leq \sum_n n^{-2} < \infty$
- $\sum_n e^{-(1/2 - |t_n|/2)\ell_n} = \sum_n e^{-\ell_n/4} = \sum_n 1/n = \infty$ then $X(4 \log n, 1/2) \in O_G$

When $t_n \equiv 1/2$, to which extent $\sum_{n=1}^{\infty} e^{-\ell_n/4} = \infty$ characterize the class $O_g$?
The non-complete half-twist surfaces

Theorem (Basmajian, Hakobyan and Š.)

Let $X(\ell_n, 1/2)$ be a half-twist flute surface with concave and increasing lengths $\ell_n$. TFAE:

- $X(\ell_n, 1/2) \in O_G$,
- $\sum_{n=1}^{\infty} e^{-\ell_n/4} = \infty$,
- $X(\ell_n, 1/2)$ is complete.

Hakobyan slice: $\ell_{2n} = a \log(n + 1) + b \log n$, $\ell_{2n+1} = (a + b) \log(n + 1)$, $t_n \equiv \frac{1}{2}$ for $a > 0$ and $b > 0$; $\ell_n$ increasing but not concave.
A Riemann surface $X$ is **symmetric** if there is an orientation reversing isometry which interchanges front to back decomposition of pairs of pants into hexagons.

**Theorem (M. Pandazis and Š.)**

Let $X_f = \mathbb{H}/\Gamma$ be a flute surface with $t_n \in \{0, \frac{1}{2}\}$ for all $n$. Then $X_f \in O_g$ if and only if $\Gamma$ is of the first kind (i.e. $X_f$ complete-no funnels or half-planes).
Symmetric surfaces

More generally, a surface is end symmetric if each component of the complement of a compact geodesic subsurface has an orientation reversing isometry whose set of fixed points divides the end into front and back.

Theorem (M. Pandazis and Š.)

Let $X = \mathbb{H}/\Gamma$ be an end symmetric Riemann surface with finitely many ends. Then $X \in \mathcal{O}_G$ iff $\Gamma$ is of the first kind.
We need to find a condition on the lengths such that the half-twist flute surface $X_f$ is complete.

for $\sigma_n = \ell_n - \ell_{n-1} + \cdots + (-1)^{n-1}\ell_1$, the expression $e^{-\sigma_n/2}$ is the length of the summit of a Sacherri quadrilateral between $\ell_n$ and $\ell_{n+1}$ obtained by concatenations.

**Proposition. (Basmajian-Hakobyan-Š.)**

Let $X = \mathbb{H}/\Gamma$ be a half-flute with cuff lengths $\{\ell_n\}$. If $\sum_{n=1}^{\infty} e^{-\sigma_n/2} < \infty$ then $\Gamma$ is of the second kind (i.e. $X$ incomplete).
In the $\Omega$-regions of Hakobyan slice, we have $\sum_{n=1}^{\infty} e^{-\sigma n/2} = \infty$.

**Theorem (Pandazis-Š.)**

Let $X$ be a half-twist flute surface in the Hakobyan slice. Then $\sum_{n=1}^{\infty} e^{-\sigma n/2} = \infty$ implies $X \in O_G$. 

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We compare $\sum_{n=1}^{\infty} e^{-\sigma n/2} = \infty$ to the length of a piecewise horocyclic path that follows a zig-zag of geodesics obtained by adding a geodesic asymptotic to $\ell_n$ and $\ell_{n+1}$ at its ends.
Integrable quadratic differentials

Analogue of Hubbard-Masur theorem for compact surfaces.

**Theorem (Š.)**

Let \( X = \mathbb{H}/\Gamma \) be an infinite Riemann surface with \( \Gamma \) of the first kind. Then the space of integrable holomorphic quadratic differentials \( A(X) \) on \( X \) is homeomorphic to a subset \( ML_f(X) \) of the space of all measured laminations on \( X \), where \( ML_f(X) \) can be realized by partial foliations on \( X \) with finite Dirichlet energy.

**Theorem (Š.)**

\( X \notin O_G \) iff there exists \( \varphi \in A(X) \) whose horizontal trajectories are escaping to \( \partial_{\infty}X \).

Brownian motion on \( X \) is recurrent iff a.e. leaf of every finite-area holomorphic quadratic differential is recurrent.

**Theorem (Š.)**

Assume \( X \in O_G \). The Teichmüller distance is given by Kerkchoff’s formula, i.e.

\[
d_T(Y, Z) = \frac{1}{2} \log \sup_{\gamma \in S} \frac{\text{ext}_Z(\gamma)}{\text{ext}_Y(\gamma)}
\]

where \( \text{ext}_Y(\cdot) \) and \( \text{ext}_Z(\cdot) \) are extremal lengths on surfaces \( Y \) and \( Z \) of the corresponding simple closed curves.
Applications to the classification of Riemann surfaces

**Theorem (Š.)**

Let $X_C$ be the Cantor tree surface with geodesic pants decomposition such that the lengths of the boundary geodesics at the level $n$ are equal to some $\ell_n$ for each $n$. If there exists $r > 2$ such that

$$\ell_n = \frac{n^r}{2^n}$$

then $X \notin O_G$.

Bridges the gap between Basmajian-Hakobyan-Š. ($\ell_n \leq n/2^n$ implies $X \in O_G$) and McMullen ($\ell_n \geq c_0 > 0$ implies $X \notin O_G$).
The proof is by constructing a finite-area holomorphic quadratic differential on $X$ whose horizontal trajectories are escaping outside $\partial_\infty X$.

Theorem (Pandazis)

Let $X_C$ be the (blooming) Cantor tree surface with geodesic pants decomposition such that there exists $r > 1$ with

$$\frac{1}{n^2} \lesssim \ell_n^j \lesssim \frac{n^r}{2^n},$$

for all $1 \leq j \leq 2^{n+1}$ then $X \notin O_G$.

The idea is to construct a partial foliation of $X_C$ using the pants decomposition and breaking each pair of pants into hexagons.
Open problems:

Problem 1. (Sullivan) A flute surface is conformal to either a complex plane minus a discrete set of points ($\in O_G$) or a disk minus a discrete set of points ($\notin O_G$). If the points accumulate to the whole $S^1$ then the surface has covering group of the first kind but it is not in $O_G$. Find the Fenchel-Nielsen coordinates of such flute surfaces.

Problem 2. (Kahn, Markovic) Given any sequence $\{a_n\}_n$ of positive numbers, is there a flute surface $X$ with cuff lengths $\ell_n = a_n$ and a choice of twists $\{t_n\}_n$ such that $X \in O_G$? Basmajian-Š. proved that there is always a choice of twists $\{t_n\}$ such that the covering group of $X$ is of the first kind, i.e. $X$ is complete.
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Thank you!